K-CONVEXITY IN $\Re^n$

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Abstract

We generalize the concept of K-convexity to an n-dimensional Euclidean space. The resulting concept of **K**-convexity is useful in addressing production and inventory problems where there are individual product setup costs and/or joint setup costs. We derive some basic properties of **K**-convex functions. We use the concept to derive the optimal policy in a deterministic case of two products with a joint setup cost. We conclude the paper with some suggestions for future research.

Keywords: K-convex, inventory models, (s, S) policy, supermodular functions

1 Introduction

One of the most important results in inventory theory is the proof of the optimality of a so-called (s, S) policy when there is a fixed cost of setup or ordering in a single-product inventory problem. The policy is characterized by two numbers s and S, $S \geq s$, such that when the inventory level falls below the level s, an order is issued for a quantity that brings the inventory up to level S, and nothing is ordered otherwise. It is customary to say that an (s, S) policy is optimal even when the two parameters vary from period to period when the problem is either finite horizon or non-stationary or both.

The idea to defer orders until the inventory level has dipped to a low enough level (s) so that the setup cost will be incurred when a large enough amount (at least $S - s$) is ordered had been appealing to researchers in the 1950s. However, its proof eluded them because the value function in the dynamic programming formulation of the problem was neither convex nor concave. Finally, Scarf [9] provided a proof by introducing a new class of functions called K-convex functions defined on $\Re^1$.

While Scarf [9] had assumed holding/backlog cost to be convex, Veinott [11], under somewhat different conditions, supplied a new proof of the optimality of the (s, S) policy. Veinott [11] assumed the negative of the one-period expected holding/backlog cost to be unimodal and (nearly) rising over time, and therefore his conditions do not imply and are not implied by conditions assumed by Scarf [9].

Since these classical works of Scarf [9] and Veinott [11], there have been a few attempts to study multiproduct extensions of the problem. For our purpose, we will only review Johnson [3], Kalin [4], Ohno and Ishigaki [7], and Liu and Esobgue [6]. Other works that we do not review are discussed by these authors.

Johnson [3] considers an n-product problem with a joint setup cost, which is incurred if one or more of the products is ordered. He uses the policy improvement method in Markov decision processes to show that the optimal policy in the stationary case is a (σ, S) policy, where $\sigma \subset \Re^n$ and $S \in \Re^n$, and one orders up to the level S if the inventory level $x \in \sigma$ and $x \leq S$ and one does not order $x \notin \sigma$. Since nothing is specified when $x \in \sigma$, $x \nleq S$, the policy is proved to be optimal only when the initial inventory level is less than or equal to S.

Kalin [4] shows that, in addition, when $x \in \sigma$ and $x \nleq S$, then there is $\bar{S}(x) \geq x$ such that the optimal policy is to order $\bar{S}(x) - x$. Such a policy can be termed a $(\sigma, S(\cdot))$ policy. Kalin [4] also characterizes the nonordering set σ^c, the complement of σ in $\Re^n$. His proof assumes a number of conditions and uses the concept called (K, η)-quasiconvexity.

Finally, Ohno and Ishigaki [7] consider a continuous-time problem with Poisson demands. They use a policy improvement method to show that the $(\sigma, S(\cdot))$ policy is optimal for their problem. They also compute the optimal policy in some cases and compare it with three well-known heuristic policies.

Among all the multiproduct models with a fixed joint setup cost, Liu and Esobgue [6] is the only one that builds on Scarf's proof. In order to accomplish this, they generalize the concept of K-convexity to $\Re^n$. They use this concept to prove the optimality of an (s_t, S_t) policy in a finite horizon case, where t denotes the time period, under the condition that the initial inventory level $x_0 \leq S_0$ and that S_t increases with t. Since this second condition cannot be verified a priori, their proof cannot be called a complete proof.

While we are not able to complete their proof, we propose a fairly general definition of **K**-convexity in $\Re^n$, which includes the cases of joint setups as well as individual setups. We then develop some properties of **K**-convex functions that we hope will lead to the solution of a general multiproduct inventory problem with different kinds of setup costs. We conclude the paper by applying the concept of **K**-convexity in solving a two-period, two-product deterministic inventory problem.

The plan of the paper is as follows. In the next section, we provide our definition of **K**-convex functions defined on $\Re^n$ and derive some properties of such functions. In Section 3, we consider the individual setups case, and show that the sum of independent **K**-convex functions is **K**-convex. In Section 4, we establish a result for supermodular **K**-convex functions. In Section 5, we consider the joint setup case and discuss the results obtained by Liu and Esobgue [6] on the optimality of a (σ, S) policy. We conclude Section 5 by using the concept of **K**-convexity to show the optimality of a (σ, S) policy in a two-period, two-product deterministic inventory problem. We conclude the paper in Section 6 by discussing some important open problems for research.

2 Definitions and Some Properties

In this section, we introduce some definitions of **K**-convexity and derive some properties of **K**-convex functions. We begin with the classical definition of **K**-convexity in the one-dimensional space given by Scarf [9].

Scarf's Definition in $\Re^1$: A function $g : \Re^1 \longrightarrow \Re^1$ is *K-convex* if

$$g(u) + z\left[\frac{g(u) - g(u-b)}{b}\right] \leq g(u+z) + K \tag{1}$$

for any u, $z \geq 0$, and $b > 0$.

Next we propose a fairly general definition of real-valued **K**-convex functions defined on $\Re^n$. Define **K**$= (K_0, K_1, \ldots, K_n)$ to be a vector of $(n+1)$ nonnegative constants. Let us define a function $K : \Re^{n+} \longrightarrow \Re^1$ as follows:

$$K(x) = K_0\delta(e'x) + \sum_{i=1}^{n} K_i\delta(x_i), \tag{2}$$

where $e = (1, 1, \ldots, 1)'^F \in \Re^n$, $\delta(0) = 0$ and $\delta(z) = 1$ for all $z > 0$.

Definition 1 *A function $g : \Re^n \to \Re$ is* **K***-convex if*

$$g(\lambda x + \bar{\lambda} y) \leq \lambda g(x) + \bar{\lambda}[g(y) + K(y - x)] \tag{3}$$

for all $x \leq y$ and all $\lambda \in [0, 1]$, where as usual $\bar{\lambda} \equiv 1 - \lambda$.

This definition is motivated by the joint replenishment problem when a setup cost K_0 is incurred whenever an item is ordered and individual setup costs are incurred for each item included in the order. There are a number of important special cases that we note in what follows.

The simplest is the case of one product or $n = 1$, where $K_0 + K_1$ can be considered to be the setup cost and (3) can be written as

$$g(\lambda x + \bar{\lambda} y) \leq \lambda g(x) + \bar{\lambda}[g(y) + K \cdot \delta(y - x)], \tag{4}$$

where $K = K_0 + K_1$. In this case, (4) is equivalent to the concept of K-convexity in $\Re$ defined by Scarf [9]; see also Denardo [1] and Porteus [8].

The next special case arises when $K_i = 0, i = 1, 2, \cdots, n$, i.e., $K = (K_0, 0, 0, \ldots, 0)$. In this case, a setup cost K_0 is incurred whenever any one or more of the products are ordered. Here

$$K(x) = K_0 \delta(e'x), \tag{5}$$

and the case is referred to as the *joint setup cost case.*

Finally, there is a case in which $K_0 = 0$. Here there is no joint setup, but there are individual setups. Thus,

$$K(x) = \sum_{i=1}^{n} K_i \delta(x_i). \tag{6}$$

The *individual setups case* will be discussed further in Section 3. The joint setup case will be treated in Section 4.

Definition (1) admits a simple geometric interpretation related to the concept of visibility, see for example Kolmogorov and Fomin [5]. Let $a \geq 0$. A point $(x, f(x))$ is said to be *visible* from $(y, f(y) + a)$ if all intermediate points $(\lambda x + \bar{\lambda} y, f(\lambda x + \bar{\lambda} y)),\ 0 \leq \lambda \leq 1$ lie below the line segment joining $(x, f(x))$ and $(y, f(y) + a)$. We can now obtain the following geometric characterization of **K**-convexity.

Theorem 2.1 *A function g is* **K***-convex if and only if $(x, g(x))$ is visible from $(y, g(y) + K(y - x))$ for all $y \geq x$.*

Proof

By **K**-convexity, the function g over the segment $\lambda x + (1 - \lambda) y,\ \lambda \in [0, 1]$, lies below the line segment joining $(x, g(x))$ and $(y, g(y) + K(y - x))$. Since $y \geq x$, $K(y - x) \geq 0$. This completes the proof.

□

Definition 2 *A function $g : \Re^n \longrightarrow \Re^1$ is* **K**-*convex if*

$$g(u) + \frac{1}{\mu}\left[g(u) - g(u - \mu h)\right] \le g(u + h) + K(h) \tag{7}$$

for every $h \in \Re^n$, $\quad h > 0, \quad \mu \in \Re, \quad \mu > 0$, and $x \in \Re^n$.

Here $h > 0$ means $h \ge 0$ and $h_i > 0$ for at least one $i \in \{1, 2, \ldots, n\}$. Note that when $h = 0$, (1) is trivially satisfied. Note that in $\Re$, this definition reduces to Scarf's definition by using the transformation $h = z$ and $\mu = z/b$.

Theorem 2.2 *Definitions 1 and 2 are equivalent.*

Proof

It is sufficient to prove that (3) can be reduced to (1) and vice versa. This can be seen by using the transformation

$$\lambda = 1/(1 + \mu), \quad x = u - \mu h, \quad y = u + h$$

and noting that

$$K(y - x) = K((1 + \mu)h) = K(h).$$

□

The following usual properties of **K**-convex functions on $\Re$ can be extended easily to $\Re^n$.

Property 1 *If $g : \Re^n \longrightarrow \Re^1$ is* **K**-*convex, then it is* **L**-*convex for any* $\mathbf{L} \ge \mathbf{K}$. *In particular, if g is convex, then it is also* **K**-*convex for any* $\mathbf{K} \ge 0$.

Proof

Since $\mathbf{L} \ge \mathbf{K}$, it follows from (3) that

$$\begin{aligned} g(\lambda x + \bar{\lambda} y) &\le \lambda g(x) + \bar{\lambda}[g(y) + K(y - x)] \\ &\le \lambda g(x) + \bar{\lambda}[g(y) + L(y - x)] \end{aligned}$$

for all $x \le y$ and all $\lambda \in [0, 1]$, where the function $L : \Re^{n+} \to \Re^1$ is given by $L(x) = L_0 \delta(e'x) + \sum_{i=1}^{n} L_i \delta(x_i)$.

□

Property 2 *If $g^1 : \Re^n \longrightarrow \Re^1$ is* **K**-*convex and $g^2 : \Re^n \longrightarrow \Re^1$ is* **L**-*convex, then for $\alpha \ge 0, \quad \beta \ge 0, \quad g = \alpha g_1 + \beta g_2$ is* $(\alpha\mathbf{K} + \beta\mathbf{L})$-*convex.*

Proof

By definition, we have

$$\begin{aligned} g^1\lambda x + \bar{\lambda} y) &\le \lambda g^1(x) + \bar{\lambda}[g^1(y) + K(y-x)], \\ g^2(\lambda x + \bar{\lambda} y) &\le \lambda g^2(x) + \bar{\lambda}[g^2(y) + L(y-x)], \end{aligned}$$

for all $x \le y$ and all $\lambda \in [0,1]$. Then

$$\begin{aligned} g(\lambda x + \bar{\lambda} y) &= \alpha g^1(\lambda x + \bar{\lambda} y) + \beta g^2(\lambda x + \bar{\lambda} y) \\ &\le \lambda[\alpha g^1(x) + \beta g^2(x)] + \bar{\lambda}[\alpha g^1(y) + \beta g^2(y) + \alpha K(y-x) + \beta L(y-x)] \\ &\le \lambda g(x) + \bar{\lambda}[g(y) + (\alpha K + \beta L)(y-x)], \end{aligned}$$

where the function $\alpha K + \beta L : \Re^{n+} \to \Re$ is defined analogously to (2).

□

Property 3 *If $g : \Re^n \longrightarrow \Re^1$ is* **K***-convex and $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$ is a random vector such that $E|g(x-\xi)| < \infty$ for all x, then $Eg(x-\xi)$ is also* **K***-convex.*

Proof

Since g is **K**-convex, we have for any $z \in \Re^n$,

$$g(\lambda(x-z) + \bar{\lambda}(y-z)) \le \lambda g(x-z) + \bar{\lambda}[g(y-z) + K(y-x)] \tag{8}$$

for all $x \le y$ and all $\lambda \in [0,1]$. Since $E|g(x-\xi)| < \infty$, we can take expectations on both sides of (8) to obtain

$$Eg(\lambda x + \bar{\lambda} y - \xi) \le \lambda Eg(x-\xi) + \bar{\lambda}[Eg(y-\xi) + K(y-x)].$$

Therefore, $Eg(x-\xi)$ is **K**-convex.

□

In addition, the following property follows immediately from the definition of **K**-convexity.

Property 4 *Let $g : \Re^n \to \Re^1$. Fix $x, y \in \Re^n$ with $x \le y$ and let*

$$f(\theta) = g(x + \theta(y-x)).$$

Then $f : \Re \to \Re$ is $K(y-x)$-convex (in the sense of Scarf [9]) if and only if g is **K***-convex.*

Proof

Assume that f is not $K(y-x)$-convex. Then there exists $\theta_1 < \theta_2$ such that

$$f(\lambda\theta_1 + \bar{\lambda}\theta_1) > \lambda f(\theta_1) + \bar{\lambda}[f(\theta_2) + K(y-x)].$$

This implies that

$$g(\lambda\tilde{x} + \bar{\lambda}\tilde{y}) > \lambda g(\tilde{x}) + \bar{\lambda}[g(\tilde{y}) + K(y-x)],$$

where $\tilde{x} = x + \theta_1(y-x)$ and $\tilde{y} = x + \theta_2(y-x) > \tilde{x}$, which contradicts the **K**-convexity of g.

Assume now that g is not **K**-convex. Then there exist x, y with $x \leq y$ such that

$$g(\lambda x + \bar{\lambda}y) > \lambda g(x) + \bar{\lambda}(g(y) + K(y-x)).$$

Let $\theta_1 = 0$ and $\theta_2 = 1$. Then the above inequality implies that

$$f(\lambda\theta_1 + \bar{\lambda}\theta_2) > \lambda f(\theta_1) + \bar{\lambda}[f(\theta_2) + K(y-x)]$$

which contradicts the $K(y-x)$-convexity of f.

□

Notice also that the function K defined in (2) satisfies the triangular inequality.

Property 5 *For all* $x \geq 0,\ y \geq 0,$ *we have*

$$K(x+y) \leq K(x) + K(y).$$

Proof
Follows from (2) and the fact that $\delta(u+v) \leq \delta(u) + \delta(v)$ for $u \geq 0,\ v \geq 0$.

□

Property 6 *For all* $x \geq 0$ *and any constant* $b > 0,\ K(bx) = K(x)$.

Proof
Follows from the fact that $\delta(bu) = \delta u$ for $u \geq 0$.

□

Theorem 2.3 *Let* $g : \Re^n \to \Re^1$ *be* **K***-convex. Let* $S \in \Re^n$ *be a finite global minimizer of* g*. Let* $x \leq S$*, and define* $f(\theta) = g(x + \theta(S-x))$*. Let* θ_1 *be any* $\theta < 1$ *such that* $f(\theta_1) = f(1) + K(S-x)$*. Then* $f(\theta)$ *is non-increasing over* $\theta < \theta_1$*, and therefore* $f(\theta) \geq f(1) + K(S-x)$ *for all* $\theta \leq \theta_1$*.*

Proof
From Property 4, $f(\theta)$ is $K(S-x)$-convex. Note from Property 5 that $K(S-x)$ depends only on the direction of the line joining S and x and not on x. Then from the standard one-dimensional case, the result follows.

□

Corollary 2.1 *Let θ_1 be as defined in Theorem 2.3. Then for any $w = x+\theta(S-x)$, $\theta \leq \theta_1$,*

$$g(w) \geq g(S) + K(S-w).$$

In the next section, we study the special case of individual setups.

3 The Individual Setups Case

The individual setups case is characterized by $K(x) = \sum_i^n \delta(x_i)$. In this case, we show that the sum of independent K-convex functions is **K**-convex.

Theorem 3.1 *Let $g_i : \Re^1 \longrightarrow \Re^1$ be K_i-convex for $i = 1, \ldots, n$. Then $g(x_1, \ldots, x_n) = \sum_{i=1}^n g_i(x_i) : \Re^n \longrightarrow \Re$ is $(0, K_1, \ldots, K_n)$-convex (i.e., **K**-convex with **K**$= (0, K_1, \ldots, K_n)$).*

Proof
Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ with $y > x$ be any two points. Define $\psi = \lambda x + \bar{\lambda} y$. See Figure 1 for the illustration of these points in $\Re^2$.

By definition, we have

$$g_i(\psi_i) \leq \lambda g_i(x_i) + \bar{\lambda}(g_i(y_i) + K_i \cdot \delta(y_i - x_i)), \tag{9}$$

Adding over i results in

$$g(\psi) \leq \lambda g(x) + \bar{\lambda}(g(y) + K(y-x)),$$

completing the proof.

□

4 K-convexity and Supermodularity

The separable property assumed in Theorem 3.1 may be quite restrictive. We extend the result by relaxing the separability to a diagonal property.

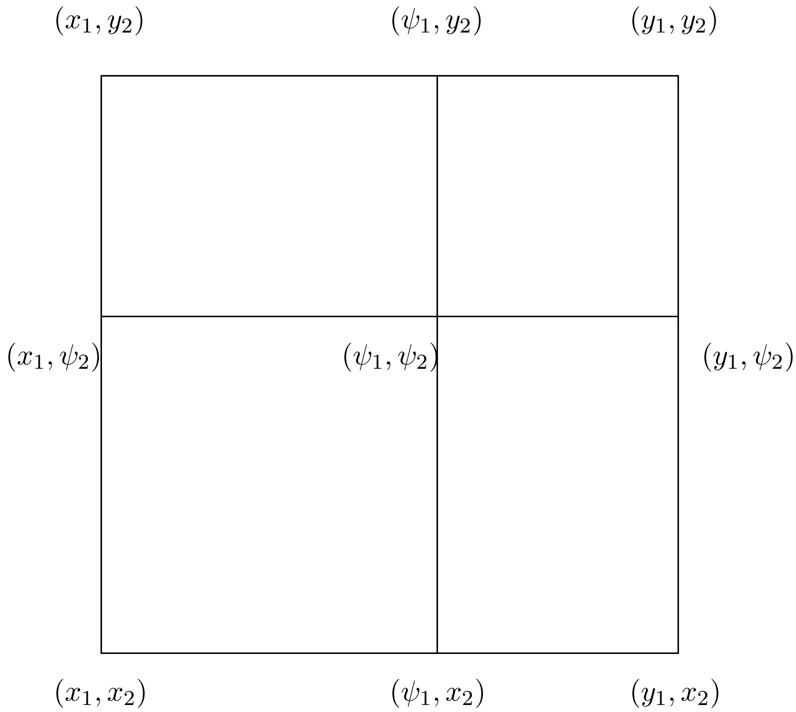

Figure 1: A grid in $\Re^2$.

A function $g : \Re^n \rightarrow \Re^1$ is *supermodular* if for any two points x and y,

$$g(x) + g(y) \leq g(x \wedge y) + g(x \vee y)$$

where the pointwise minimum of x and y is called the *meet* of x and y and is denoted by

$$x \wedge y = (x_1 \wedge y_1, x_2 \wedge y_2, \ldots, x_n \wedge y_n)$$

and the pointwise maximum of x and y is called the *join* of x and y and is denoted by

$$x \vee y = (x_1 \vee y_1, x_2 \vee y_2, \ldots, x_n \vee y_n).$$

The reader is referred to Topkis [10] for a general definition of a supermodular function defined on a lattice.

Let $g : \Re^n \longrightarrow \Re^1$. For each $z \in \Re^n$ and for each subset $I \subset N \equiv \{1, \ldots, n\}$, let $g_I(\cdot; z) : \Re^{|I|} \rightarrow \Re$ be the function that results by freezing the components of z that are not in I. The domain of the function g_I are vectors of the form $\sum_{i \in I} x_i e_i$ where e_i is the ith unit vector. Sometimes we will abuse notation and write $g_I(z)$ as a shorthand for $g_I(\cdot; z)$ and $g_i(z)$ instead of the more cumbersome $g_{\{i\}}(\cdot; z)$. Then, $g_\emptyset(z)$ is just the constant $g(z)$, while $g_i(z) = g(z_1, \ldots, z_{i-1}, \cdot, z_{i+1}, \ldots, z_n)$.

Let $K : \Re^n_+ \rightarrow \Re^1_+$ be a function with the property

$$K_{I \cup k}(x_{I \cup k}) \geq K_I(x_I) + K_k(x_k) \tag{10}$$

for all $k \notin I,\ I \neq N$.

A function g is said to be K_I-convex if for all $x, y \in \Re^{|I|}$, $x \leq y$, and for all $z \in \Re^n$, we have

$$g_I(\lambda x + \bar{\lambda} y; z) \leq \lambda g_I(x; z) + \bar{\lambda}(g_I(y; z) + K_I(y - x))$$

for all $\lambda \in (0, 1)$.

Theorem 4.1 *If g is supermodular and g is K_i-convex for all $i \in \{1, \ldots, n\}$, then g is K_I-convex for all $I \subset \{1, \ldots, n\}$.*

Proof

The proof is by induction in the cardinality of I. By hypothesis, the result holds for $|I| = 1$. Assume that g is K_I-convex for all $|I| \leq m$ for some $m < n$. We will show that the result holds for $m + 1$.

For this purpose, consider a subset of $\{1, \ldots, n\}$ of $m + 1$ distinct components. We can write this set at $J = I \cup k$ for some $k \notin I$ and an a set I of cardinality m. Let x and y be any two vectors in $\Re^{m+1}$. To show that g is K_J-convex we need to show that

$$g_J(\lambda x + \bar{\lambda} y; z) \leq \lambda g_J(x; z) + \bar{\lambda}(g_J(y; z) + K_J(y - x)).$$

for all $x, y \in \Re^{m+1}$ such that $x \leq y$.

Let $\psi = \lambda x + \bar{\lambda} y$ and $\tilde{\Delta} = y - x$ so we can write

$$\psi = x + \Delta$$

where $\Delta = \bar{\lambda}\tilde{\Delta}$.

Let Δ_I be the vector that results from Δ by making zero the component corresponding to item k and let Δ_k be the vector that results from Δ by making zero the components corresponding to items in I. Similar definitions apply to the vectors $\tilde{\Delta}_I$ and $\tilde{\Delta}_k$.

Consider the vectors: $x + \Delta_I$ and $x + \Delta_I + \tilde{\Delta}_k$. Clearly

$$\begin{aligned} \psi &= x + \Delta \\ &= x + \Delta_{I \cup k} \\ &= x + \Delta_I + \bar{\lambda}\tilde{\Delta}_k \\ &= \lambda(x + \Delta_I) + \bar{\lambda}(x + \Delta_I + \tilde{\Delta}_k). \end{aligned}$$

Consequently,

$$g_J(\psi; z) \leq \lambda g_J(x + \Delta_I; z) + \bar{\lambda}[g_J(x + \Delta_I + \tilde{\Delta}_k; z) + K_k(\tilde{\Delta}_k)]. \tag{11}$$

Now consider the vectors $x + \Delta_k$ and $x + \Delta_k + \tilde{\Delta}_I$. Once again,

$$\begin{aligned} \psi &= x + \Delta \\ &= x + \Delta_{I \cup k} \\ &= x + \Delta_I + \bar{\lambda}\tilde{\Delta}_k \\ &= \lambda(x + \Delta_k) + \bar{\lambda}(x + \Delta_k + \tilde{\Delta}_I). \end{aligned}$$

Consequently,

$$g_J(\psi; z) \leq \lambda g_J(x + \Delta_k; z) + \bar{\lambda}[g_J(x + \Delta_k + \tilde{\Delta}_I; z) + K_I(\tilde{\Delta}_I)]. \tag{12}$$

Adding the right hand size of inequalities (11) and (12) and using the supermodularity property we obtain the inequalities

$$g_J(x + \Delta_I; z) + g_J(x + \Delta_k; z) \leq g_J(x; z) + g_J(x + \Delta; z) = g_J(x; z) + g_J(\psi; z)$$

and

$$g_J(x + \Delta_I + \tilde{\Delta}_k; z) + g_J(x + \Delta_k + \tilde{\Delta}_I; z) \leq g_J(\psi; z) + g_J(y; z))$$

we obtain

$$2g_J(\psi; z) \leq \lambda[g_J(x; z) + g_J(\psi; z)] + \bar{\lambda}[g_J(\psi; z) + g_J(y; z) + K_k(\tilde{\Delta}_k) + K_I(\tilde{\Delta}_I)],$$

or equivalently,

$$g_J(\psi; z) \leq \lambda g_J(x; z) + \bar{\lambda}(g_J(y; z) + K_k(\tilde{\Delta}_k) + K_I(\tilde{\Delta}_I)].$$

Finally, using the fact that $K_k(\tilde{\Delta}_k) + K_I(\tilde{\Delta}_I) = K_k(\tilde{\Delta}) + K_I(\tilde{\Delta}) \leq K_J(\tilde{\Delta})$ establishes that g is J-convex.

Since I and k were arbitrary this implies that g is K_I-convex for all subsets of cardinality $|J| = m + 1$, and therefore for all subsets of $\{1, \ldots, n\}$.

□

In the next section, we provide two examples of K that satisfy the property (10).

4.1 Examples of K Satisfying (10)

One trivial example is the case where $K(x) = \sum_{j=1}^n K_j\delta(x_j)$ with δ denoting an indicator function such that $\delta(x_j) = 1$ if $x_j > 0$ and zero otherwise. Thus, the result here implies the result we obtained before under the weaker assumption of pairwise supermodularity. The result is actually somewhat stronger because it holds for all subsets I not only for the full set $I = \{1, \ldots, n\}$.

While the definition of K does not allow for the traditional joint replenishment function $K(x) = \sum_{j=1}^n K_j\delta(x_j) + K_0\delta(\sum_j x_j)$, it does allow other forms of joint setup costs. For example, let $K(x) = \sum_{j=1}^n K_j\delta(x_j)$ if at most one of the components are positive and equal to $K(x) = \sum_{j=1}^n K_j\delta(x_j) + K_0$ if two or more of the components are ordered.

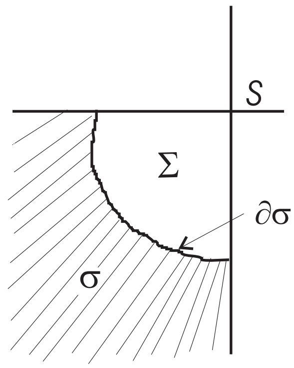

Figure 2: Minimum Point $S, \Sigma, \partial\sigma$, and σ.

5 The Joint Setup Case

In the joint setup case, we have $K(x)$ as defined in (5). Thus, we can rewrite (3) as

$$g(\lambda x + \bar{\lambda} y) \leq \lambda g(x) + [g(y) + K_0 \delta(e'(y - x))] \tag{13}$$

in our definition of **K**-convexity in the joint setup case.

For ease of exposition, we shall refer to this special case as $\mathbf{K_0}$-convexity. We also note that Definition 2 reduces to the definition of $\mathbf{K_0}$-convexity proposed by Liu and Esogbue [6].

Consider a continuous **K**-convex function g, which is coercive, i.e., $\lim_{\|x\| \to \infty} g(x) = \infty$. Then it has a global minimum point S. Define the set $\partial\sigma$ as follows:

$$\partial\sigma = \{x \leq S \mid g(x) = g(s) + K_0\}. \tag{14}$$

Clearly, $\partial\sigma$ is non-empty and bounded because of the coercivity of $g(x)$. Now define the following two regions:

$$\Sigma = \{x \leq S \mid \exists s \in \partial\sigma \text{ such that } x \in \overline{sS}\}, \tag{15}$$

where $\overline{sS}$ is the line segment joining s and S. Clearly, $\partial\sigma \subset \Sigma$. The second region is

$$\sigma = \{x \leq S \mid x \notin \Sigma\}. \tag{16}$$

Note that $\sigma \bigcap \Sigma = \sigma \bigcap \partial\sigma = \emptyset$. See Figure 2.

Lemma 5.1 *If g is $\mathbf{K_0}$-convex, continuous and coercive, then*

$$\begin{aligned} i) x \in \Sigma \quad &\Rightarrow \quad g(x) \leq K_0 + g(S), \\ ii) x \in \sigma \quad &\Rightarrow \quad g(x) > K_0 + g(S). \end{aligned}$$

Proof

(i) Suppose there is an $x \in \Sigma$ such that $g(x) > K_0 + g(S)$. By the definition of Σ, there is an S such that $x \in \overline{sS}$ and $g(s) = K_0 + g(S)$. Thus, $g(x) > g(s)$, and s is not visible from $K_0 + g(s)$, contradicting the **K**-convexity of g.

(ii) Follows from that fact that $\sigma \bigcap \Sigma = \emptyset$.

$\square$

Lemma 5.2 *If g is continuous $\mathbf{K_0}$-convex, then*

$$\begin{aligned} f(x) &= \inf_{y \geq x} \{g(y) + K_0 \delta(e'(y-x))\} \\ &= \begin{cases} K_0 + g(S), & x \in \sigma \\ g(x), & x \in \Sigma \end{cases}, \end{aligned} \tag{17}$$

where S is the global minimum of g. Furthermore, f is continuous and $\mathbf{K_0}$-convex on $\{x \leq S\}$.

Proof

For $x \in \sigma$, we have $g(x) > K + g(S)$. Since $y = S$ is feasible and since $K + g(S) \leq K + g(y)$ for all $y < S$, we can conclude that the minimum is $y^*(x) = S$ and, therefore, $f(x) = K_0 + g(S)$.

For $x \in \Sigma$, $g(x) \leq K_0 + g(S)$. Then for any $y > x$, $g(x) \leq K_0 + g(S) \leq K_0 + g(y)$. Thus, $y^*(x) = x$ is the minimum and $f(x) = g(x)$.

To prove that f is continuous, it is sufficient to prove its continuity on any closed set $\Gamma \subset \Re^n$. By the assumption on g, we know that the set $\Lambda = \{y^*(x) \mid x \in \Gamma\}$ of optimal values $y^*(x),\ \ x \in \Gamma$, is compact. Then, $g(x)$ is uniformly continuous on Λ. That is, for any $\epsilon > 0$, there is a $\delta > 0$ such that $\mid g(y_1) - g(y_2) \mid < \epsilon$ if $y_1,\ y_2 \in \Lambda$ and $\|y_1 - y_2\| < \delta$. Now let any two points $x_1, x_2 \in \Gamma$ such that $\|x_1 - x_2\| < \delta$. Let $y^*(x_1)$ and $y^*(x_2)$ denote the solutions, respectively.

Consider the two cases $y^*(x_1) > x_1$ and $y^*(x_1) = x_1$. In the first case, $y^*(x_1) = S$ and there exists $y_2 \geq x_2$ such that $\|S - y_2\| < \delta$. By this and (17), we have

$$f(x_1) = K_0 + g(S) \geq K_0 + g(y_2) - \epsilon \geq f(x_2) - \epsilon. \tag{18}$$

Likewise, in the second case

$$f(x_1) = g(x_1) \geq g(x_2) - \epsilon \geq f(x_2) - \epsilon. \tag{19}$$

Thus, we have $f(x_1) \geq f(x_2) - \epsilon$ in both cases. Similarly, we can prove $f(x_2) \geq f(x_1) - \epsilon$. This completes the proof of the continuity of f.

Next we prove that f is $\mathbf{K_0}$-convex on $\{x \leq S\}$ by using Theorem 2.1. We need to examine three cases of $(x, y), y \geq x$.

(i) $x \in \sigma$, $y \in \sigma$

(ii) $x \in \Sigma$, $y \in \Sigma$

(iii) $x \in \sigma$, $y \in \Sigma$.

In (i), $f(x) = f(y) = K + g(S)$ and so f is $\mathbf{K_0}$-convex. In (ii), $f = g$ and g is convex, so f is convex. In (iii), $K_0 + g(y) \geq K_0 + g(S) = f(x)$, and from Lemma 5.1,

$$f(z) = g(z) \leq K_0 + g(S) \tag{20}$$

for every $z \in [x, y]$. So every $(x, f(z))$ is visible from $(y + f(y))$. This completes the proof of $\mathbf{K_0}$-convexity of function f.

□

Lemma 5.2 provides the essential induction step in a proof of an (s, S) type policy. The following result, which has been proved by Kalin [4] without using $\mathbf{K}$-convexity, is expected to be derived with the use of Lemma 5.2.

Theorem 5.1 *Assume $g(x)$ representing the cost of inventory/backlog is convex, continuous and coercive. Let K_0 represent the joint setup cost of ordering. Then there is a value S and partition of the region $\{x \leq S\}$ into disjoint sets σ and Σ such that given an initial inventory $x_0 \leq S$, the optimal ordering policy for the infinite horizon inventory problem is*

$$y^*(x) = \begin{cases} S, & \text{if } x \in \sigma, \\ x, & \text{if } x \in \Sigma. \end{cases}$$

A finite horizon version of Theorem 5.1 was proved by Liu and Esobgue [6] under a condition that the order-up-to-levels in different periods increase over time. But this condition is not convenient, as it cannot be verified a priori.

5.1 A Deterministic Example

Iyer [2] has analyzed the deterministic (s, S) inventory problem for a single product. We consider a deterministic two-period, two-product inventory problem with the inventory/backlog cost $\lambda(x_1, x_2) =| x_1 | + | x_2 |$, the first period demand (1,1), the second period demand

(4,4), and the joint setup cost $K_0 = 2$. Because we allow backlogs, all demands have to met. Therefore, we can ignore the variable purchase cost of the units ordered. We will set the unit purchase cost to zero without loss of generality. Our purpose is to find the optimal ordering policy in each period to minimize the total holding and backlog costs over the two period horizon.

Let $g_t(x)$ be the optimal cost to go with t periods remaining, $t = 0, 1, 2$. Clearly, since all demand must be satisfied, we could define

$$g_0(x) = \begin{cases} 0, & x \geq (4,4), \\ \infty, & \text{otherwise}, \end{cases}$$

and we will have

$$g_1(x_1, x_2) = \inf_{y \geq x} [2\delta(e'(y-x)) + g_0(y)] = \begin{cases} 2, & x \in \sigma_1, \\ 2+ \mid x_2 - 4 \mid, & x \in \sigma_1^1, \\ 2+ \mid x_1 - 4 \mid, & x \in \sigma_1^2, \\ \mid x_1 - 4 \mid + \mid x_2 - 4 \mid, & x \in \sigma_1^0, \\ 0, & x = \sigma_1^*. \end{cases} \tag{21}$$

where

$$\begin{aligned} \sigma_1 &= \{x \mid x \leq 4, x_2 < 4, x \neq (4,4)\}, \\ \sigma_1^1 &= \{x \mid x_1 \leq 4, x_2 > 4\}, \\ \sigma_1^2 &= \{x \mid x_1 > 4, x_2 \leq 4\}, \\ \sigma_1^0 &= \{x \mid x_1 > 4, x_2 > 4\}, \\ \sigma_1^* &= (4,4). \end{aligned} \tag{22}$$

In Figure 3 , we show the regions $\sigma_1, \sigma_1^1, \sigma_1^2, \sigma_1^0$ and σ_1^*.

Theorem 5.2 *$g_1(x_1, x_2)$ is (2,0,0)-convex. It attains its global minimum at $S_1 = (4,4)$. Finally, $g(\cdot, x_2)$ is nondecreasing in x_1 for $x_1 \geq 4$ and $g(x_1 \cdot)$ is nondecreasing in x_2 for $x_2 \geq 4$.*

Proof

Note that $g_1(x_1, x_2) \geq 0$ and $g_1(4,4) = 0$. Thus, $S_1 = (4,4)$ is a global minimum of $g_1(x_1, x_2)$. To prove that $g_1(x_1, x_2)$ is (2,0,0)-convex, let x and $y \geq x$ be two arbitrary points in the (x_1, x_2) space. Let us choose x and y as shown in Figure 3 . Let us define

$$f_{xy}(\theta) = g(x + \theta(y - x)). \tag{23}$$

It is clear from Figure 4 that for θ_1 and $\theta_2 \geq \theta_1$, $(\theta_1, f(\theta_1))$ is visible from $(\theta_2, f(\theta_2) + 2)$. Thus, $f_{xy}(\theta)$ is 2-convex. We can repeat this argument for any other pair x and $y, y \geq x$. By Theorem 2.1, therefore, $g_1(x_1, x_2)$ is (2,0,0)-convex.

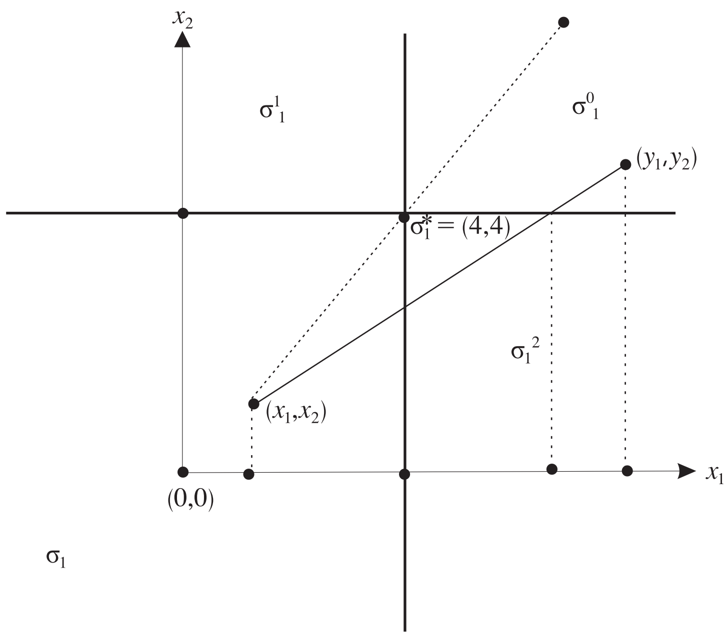

Figure 3: Ordering and no-ordering regions for $t = 1$.

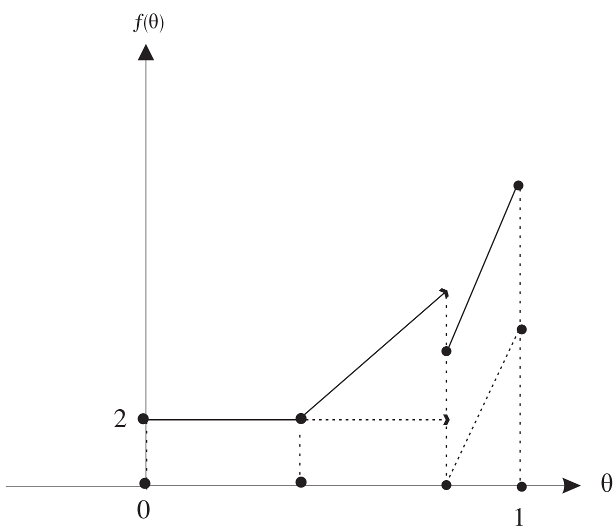

Figure 4: The value function $f_{xy}(\theta)$ along the direction $\overrightarrow{xy}$.

Finally, the increasing property of $g_1(\cdot, x_2)$ for $x_1 \geq 4$ and of $g_1(x_1, \cdot)$ for $x_2 \geq 4$ is obvious from (21). This completes the proof.

$\square$

In Figure 3, we have also shown another direction passing through (4,4) as a dotted line. Note that the function $f_{x\sigma_1}(\theta)$ along this direction takes the global minimum for $g_1(x_1, x_2)$ at (4,4).

We now proceed to examine $g_2(x_1, x_2)$. From dynamic programming, we have

$$g_2(x_1, x_2) = \inf_{\substack{y \geq x \\ y \geq (4,4)}} [2\delta(e'(y - x))+ \mid y_1 - 1 \mid + \mid y_2 - 1 \mid + g_1(y_1 - 1, y_2 - 1)]. \tag{24}$$

First, we observe that $g_2(5,5) =\mid 5-1 \mid + \mid 5-1 \mid= 8$. Also for $x_1 > 5$ and $x_2 \leq 5$, we have $g_2(x_1, x_2) = g_2(5, x_2) + 2 \mid x_1 - 5 \mid$. This is because x_1 is sufficient to satisfy the demands for both periods, and therefore there is no need to order product 1. The cost of the excess inventory $x_1 - 5$ in two periods is $2 \mid x_1 - 5 \mid$. Likewise, for $x_2 > 5$ and $x_1 \leq 5$, $g_2(x_1, x_2) = g_2(x_1, 5) + 2 \mid x_2 - 5 \mid$, and for $x_1 > 5$ and $x_2 > 5$, we have $g_2(x_1, x_2) = g_2(5,5) + 2 \mid x_1 - 5 \mid + 2 \mid x_2 - 5 \mid= 8 + 2 \mid x_1 - 5 \mid + 2 \mid x_2 - 5 \mid$. Thus, it is sufficient to consider the initial $x = (x_1, x_2)$ to satisfy $x_1 \leq 5$ and $x_2 \leq 5$.

To obtain $g_2(x_1, x_2)$ for $x_1 \leq 5$, $x_2 \leq 5$, let us consider the following four regions depicted in Figure 5. These are

$$\begin{aligned}
\sigma_2^1 &= \{x_1 \leq -1,\ 1 < x_2 \leq 5\},\ \sigma_2^2 = \{x_2 \leq -1,\ 1 < x_1 \leq 5\}, \\
\sigma_2 &= \{x_1 \leq 1,\ x_2 \leq 1,\ x_1 + x_2 \leq 0\},\ \sigma_2^* = \{(5,5)\}, \\
\sigma_2^0 &= \{x_1 \leq 5,\ x_2 \leq 5\} \setminus (\sigma_2^1 \cup \sigma_2^2 \cup \sigma_2 \cup \sigma_2^*) = \{x_1 > -1, x_2 > -1, x_1 + x_2 > 0, x \neq (5,5)\}.
\end{aligned}$$

For $x \in \sigma_2^* = (5,5)$, clearly, $g_2(5,5) =\mid 5-1 \mid + \mid 5-1 \mid= 8$.

Consider $x \in \sigma_2$. Then for $y \geq x$, $(y_1 - 1, y_2 - 1)$ can be in $\sigma_1, \sigma_1^1, \sigma_1^2$ or σ_1^0. From (21) it is clear that for any $(y_1 - 1, y_2 - 1)$ in σ_1^1, σ_1^2 or σ_1^0, one can do better by a $(y_1 - 1, y_2 - 1)$ in σ_1 as depicted in Figure 3. For any $(y_1 - 1, y_2 - 1) \in \sigma_1$, we have

$$g_2(x_1, x_2) = \min_{y \geq x}\{2+ \mid y_2 - 1 \mid + \mid y_2 - 1 \mid +2\}.$$

The minimum of the right-hand side occurs at $S_2 = (1,1)$, and therefore, $g_2(x_1, x_2) = 4$ for $x \in \sigma_2$. Note that for x on the line segment AB, $y = x$ also provides the minimum. These are the inventory/backlog levels for which we are indifferent between not ordering and ordering to (1,1). We have chosen in our formulation to order when x is on the line segment AB.

Now consider $x \in \sigma_2^1$. For $y \geq x$, we have $(y_1 - 1, y_2 - 1) \in \sigma_1 \cup \sigma_1^*$. Note that we only consider $y \leq (5,5)$ as discussed earlier. If $(y_1 - 1, y_2 - 1) \in \sigma_1^*$, then the cost is

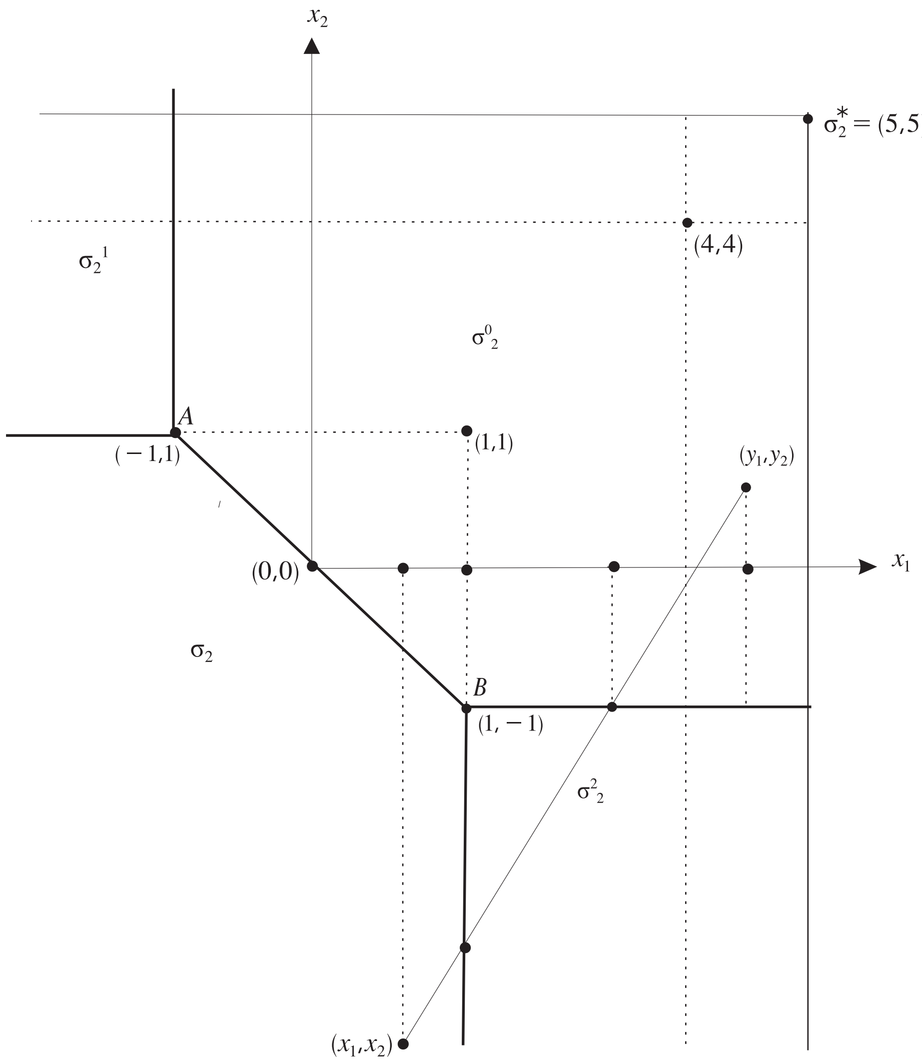

Figure 5: Ordering and no-ordering regions for $t = 2$.

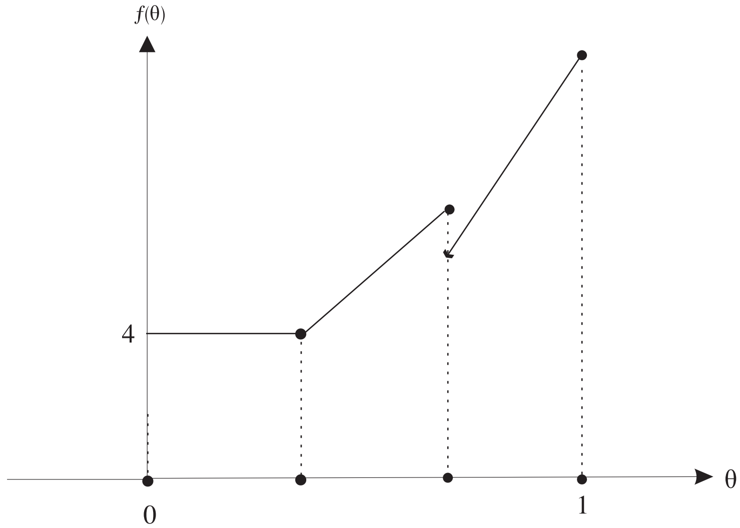

Figure 6: The value function $f_2(\theta)$.

$2+ \mid 5-1 \mid + \mid 5-1 \mid +0 = 10$. Thus,

$$\begin{aligned} g_2(x_1,x_2) &= \min \begin{cases} 10 \\ \min_{\substack{y \geq x \\ (y_1-1,y_2-1)\in\sigma_1}} \{2\delta(e'(y-x))+ \mid y_1-1 \mid + \mid y_2-1 \mid +2\} \end{cases} \\ &= \min \begin{cases} 10 \\ 2+(x_2-1)+2 \qquad \text{[order to } (1,x_2)] \end{cases} \\ &= 4+ \mid x_2-1 \mid . \qquad \text{[since } x_2 \leq 5] \end{aligned}$$

Similarly for $x \in \sigma_2^2$, we have $g_2(x_1,x_2) = 4+ \mid x_1-1 \mid$.

Finally, when $x \in \sigma_2^0$, then $(y_1-1,y_2-1) \in \sigma_1$. Thus,

$$\begin{aligned} g_2(x_1,x_2) &= \min \begin{cases} \mid x_1-1 \mid + \mid x_2-1 \mid +2 \qquad \text{[do not order]} \\ 2+4+4 \qquad \text{[order to (5,5)]} \end{cases} \\ &= 2+ \mid x_1-1 \mid + \mid x_2-1 \mid . \end{aligned}$$

Let us recapitulate below the value function $g_2(x)$:

$$g_2(x_1,x_2) = \begin{cases} 8, & x \in \sigma_2^*, \\ 4, & x \in \sigma_2, \\ 4+ \mid x_2-1 \mid, & x \in \sigma_2^1, \\ 4+ \mid x_1-1 \mid, & x \in \sigma_2^2, \\ 2+ \mid x_1-1 \mid + \mid x_2-1 \mid & x \in \sigma_2^0. \end{cases} \tag{25}$$

Theorem 5.3 *$g_2(x_1,x_2)$ is (2,0,0)-convex. Its global minimum is attained at $S_2 = (1,1)$. Finally, $g_2(\cdot,x_2)$ is increasing in x_1 for $x_1 \geq 1$ and $g(x_1,\cdot)$ is increasing in x_2 for $x_2 \geq 1$.*

Proof
Note that $g_2(x_1, x_2) \geq 2$ and $g_2(1,1) = 2$. Thus, $S_2 = (1,1)$ is a global minimum of $g_2(x_1, x_2)$. To prove that $g_2(x_1, x_2)$ is (2,0,0)-convex, first consider x and $5 \geq y \geq x$ be two arbitrary points in the (x_1, x_2)-space. Let us choose x and y as shown in Figure 5. Let us define

$$f_2(\theta) = g(x + \theta(y - x)). \tag{26}$$

This function is shown in Figure 6. By the visibility argument, this function is clearly 2-convex. We can repeat this argument for any other pair x and y, $5 \geq y \geq x$. Furthermore, since we have defined $g_2(x)$ also in regions $\{x_1 > 5,\ x_2 \leq 5\}$, $\{x_1 \leq 5,\ x_2 > 5\}$, and $\{x_1 > 5,\ x_2 > 5\}$, we can also verify similarly that the functions defined in (26) are 2-convex for any x and y, $y \geq x$. Thus, $g_2(x)$ is (2,0,0)-convex.

Finally, the increasing property for $g_2(\cdot, x_2)$ for $x_1 \geq 1$ and of $g_2(x_1, \cdot)$ for $x_2 \geq 1$ is obvious from (25). This complete the proof.

□

Corollary 5.1 *With two periods remaining, the optimal ordering policy is as follows:*

- *order up to (1,1) when* $x \in \sigma_2$
- *order up to (1,*x_2*) when* $x \in \sigma_2^1$
- *order up to (*x_1*,1) when* $x \in \sigma_2^2$
- *order nothing when* $x \in \sigma_2^0 \cup \sigma_2^*$.

6 Future Research and Concluding Remarks

In applications, it will be important to find conditions under which the **K**-convexity persists after a dynamic programming iteration. The key question is the following. If g is **K**-convex and

$$f(x) = \inf_{z \geq x}\{g(z) + K(z - x)\},$$

then is f **K**-convex? This is certainly true in $\Re^1$. For $\Re^n$, $n > 1$, things get complicated unless we impose additional conditions such as monotone order-up to levels. One direction for future research is to find out conditions that would guarantee the **K**-convexity of f. Another direction would be to further qualify **K**-convexity in some appropriate way, and then prove that this qualified **K**-convexity is preserved after a dynamic programming iteration.

In this note we have defined the notion of **K**-convexity in $\Re^n$ and have developed properties of **K**-convex functions. We have shown that g is **K**-convexity in $\Re^n$ if $g = \sum_i g_i$ and each g_i is K_i-convex in $\Re_1$ and if g is supermodular and K_I convex for each $I \subset \{1, \dots, n\}$. We have also reviewed the literature on multi-product inventory problems with fixed costs and presented a deterministic example where **K**-convexity is preserved.

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References

[1] Denardo, E. V. (1982) *Dynamic Programming, Models and Applications*, Prentice Hall, Inc., Inglewood Cliffs, NJ.

[2] Iyer, A. (1992) "Analysis of the Deterministic (s, S) Inventory Problem," *Management Science*, 38, 9, 1299-1313.

[3] Johnson, E. L. (1967) "Optimality and Computation of (σ, S) Policies in the Multi-item Infinite Horizon Inventory Problem," *Management Science*, 13, 7, 475-491.

[4] Kalin, D. (1980) "On the Optimality of (σ, S) Policies," *Mathematics of Operations Research*, 5, 293-307.

[5] Kolmogorov, A. N., and Fomin, S. V. (1970) *Introduction to Real Analysis*, Dover Publications Inc, New York.

[6] Liu, B., and Esogbue, A. O. (1999) *Decision Criteria and Optimal Inventory Processes*, Kluwer, Boston, MA.

[7] Ohno, K. and Ishigaki, T. (2001) "A Multi-item Continuous Review Inventory System with Compound Poisson Demands," *Mathematical Methods in Operations Research*, 53, 147-165.

[8] Porteus, E. (1971) "On the Optimality of Generalized (s, S) Policies," *Management Science*, 17, 411-426.

[9] Scarf, H. (1960) "The Optimality of (S, s) Policies in the Dynamic Inventory Problem," Chapter 13 in K.J. Arrow, S. Karlin, and P. Suppes (Eds.), *Mathematical Methods in Social Sciences*, Stanford University Press, Stanford, CA.

[10] Topkis, D. M. (1978) "Minimizing a Submodular Function on a Lattice," *Operations Research*, 26, 305-321.

[11] Veinott, Jr., A. F. (1966) "On the Optimality of (s, S) Inventory Policies: New Conditions and a New Proof," *SIAM Journal of Applied Mathematics*, 14, 1067-1083.